Value-at-Risk & Tail Value-at-Risk

Chelsea Anne Go, Brian Godwin Lim

Contents

1 Introduction

Value-at-Risk (VaR) is a statistical measure of the maximum potential loss of a portfolio of instruments over a given time horizon and a given confidence level in a normal market environment. It is commonly used to evaluate or measure risk for trading portfolios. VaR is a function of the time horizon and the confidence level. In the local banking industry, the VaR of a trading portfolio is usually reported daily at a 99% confidence level and for a time horizon from 1 day to 10 days. Moreover, there are multiple methods of computing for the VaR such as the delta-normal approach (or variance-covariance approach), the historical approach, and the Monte Carlo simulation approach. The delta-normal approach and historical simulation approach are more commonly used by local banks. The delta-normal approach assumes that the returns of risk factors follow a normal distribution with a mean of 0 and a standard deviation estimated from historical data. In the historical simulation approach, the returns are assumed to replicate historical returns. Additionally, the Monte Carlo simulation approach is not as commonly used since it generates random numbers from the distribution of the risk factors to obtain possible values of the change in the value of the portfolio.

Mathematically, VaR describes the quantile of the distribution of the change in the value of a portfolio, both gains and losses, over a time horizon. Suppose the time horizon is T days and the confidence level is $(100c)\%$, where $c \in [0, 1]$. If $\Delta V = V_T - V_0$ is the change in the value of the portfolio over the next T days, then the T-day $(100c)\%$ VaR is the absolute value of the $100(1-c)$th percentile of the distribution of ΔV. Denoting $|X|$ as the VaR,

$$\mathbb{P}\left(\Delta V \geq X\right) = c.$$

It is worth noting that since the value of V_T is unknown today, ΔV is considered a random variable. Additionally, VaR is consistently reported as a positive number within and across portfolios.

2 VaR for FX Spot Portfolio

2.1 Conceptual Discussion

2.1.1 Single-Asset Portfolio

Consider an FX spot position in L units of the base currency with a specified quote currency. Let x_0 be the exchange rate as of position date (today) from the base currency to the quote currency. The value of the FX spot position today in terms of the quote currency is

$$V_0 = \omega \cdot L \cdot x_0,$$

where $\omega = 1$ indicates a long position for the base currency while $\omega = -1$ indicates a short position for the base currency.

The value of the FX spot position tomorrow is then

$$V_1 = \omega \cdot L \cdot x_1,$$

where x_1 is the exchange rate tomorrow from the base currency to the quote currency. The value of V_1 is unknown today since the exchange rate tomorrow is unknown.

Thus, the change in the value of the FX spot position is

$$\begin{aligned}\Delta V &= V_1 - V_0 \\ &= \omega \cdot L \cdot (x_1 - x_0) \\ &= \omega \cdot L \cdot x_0 \cdot \frac{x_1 - x_0}{x_0},\end{aligned}$$

where $\frac{x_1 - x_0}{x_0}$ is the percentage return. Furthermore, using Taylor's series expansion,

$$\frac{x_1 - x_0}{x_0} \approx \ln\left(\frac{x_1}{x_0}\right).$$

Let $R = \ln\left(\frac{x_1}{x_0}\right)$. ΔV can then be written as

$$\Delta V = \omega \cdot L \cdot x_0 \cdot R.$$

Therefore, VaR can be obtained from ΔV, where its distribution is dependent on the distribution of R. In order to obtain the distribution of R, historical data are considered.

Let y_j be the exchange rate from the base currency to the quote currency j days ago where $j = 0, 1, \ldots, N$. In the delta-normal approach, the return R is assumed to be normally distributed $R \sim N(0, \sigma^2)$, where σ is the standard deviation of the historical returns $\{R_j\}$. R_j is the continuously compounded return obtained from the exchange rates j days and $j+1$ days ago, where $R_j = \ln\left(\frac{y_j}{y_{j+1}}\right)$. Suppose $\omega = 1$. Denote the 1-day 99% VaR by $|X|$.

$$\begin{aligned}\mathbb{P}(\Delta V \geq X) &= 0.99 \\ \mathbb{P}(\omega \cdot L \cdot x_0 \cdot R \geq X) &= 0.99 \\ \mathbb{P}\left(\frac{R}{\sigma} \geq \frac{X}{\omega \cdot L \cdot x_0 \cdot \sigma}\right) &= 0.99 \\ \mathbb{P}\left(\frac{R}{\sigma} \leq \frac{X}{\omega \cdot L \cdot x_0 \cdot \sigma}\right) &= 0.01 \\ F\left(\frac{X}{\omega \cdot L \cdot x_0 \cdot \sigma}\right) &= 0.01,\end{aligned}$$

where $F(\cdot)$ is the cumulative distribution function (cdf) of the standard normal distribution. Thus, the 1-day 99% VaR is

$$\begin{aligned}X &= \omega \cdot L \cdot x_0 \cdot \sigma \cdot F^{-1}(0.01) \\ |X| &= L \cdot x_0 \cdot \sigma \cdot F^{-1}(0.99).\end{aligned}$$

By the symmetry of the normal distribution, the same formula holds when $\omega = -1$.

2.1.2 Multi-Asset Portfolio

The Value-at-Risk of a portfolio can be obtained in two ways: the undiversified VaR and the diversified VaR. First, the undiversified VaR of a portfolio is calculated by simply adding the individual VaR amounts of each of the asset in the portfolio. Mathematically,

$$\text{VaR}_{\text{Portfolio}} = \sum_{i=1}^{n} \text{VaR}_i,$$

where VaR_i is the VaR of the ith asset in the portfolio. However, this method does not take into consideration the correlation of the risk factors of the different assets in the portfolio.

On the other hand, the diversified VaR of a portfolio considers the correlation of the risk factors of the different assets in the portfolio to reflect the benefits of diversification. The latter is always less than or equal to the undiversified VaR. In practice, banks calculate for both the undiversified and diversified VaR.

Consider a portfolio of FX spot positions in L_i units of the ith base currency where $i = 1, 2, \ldots, n$. Let $x_{i,0}$ be the exchange rate as of position date (today) from the ith base currency to the quote currency and ω_i indicate the (long/short) position for the ith base currency. The value of the portfolio today is

$$V_0 = \sum_{i=1}^{n} \omega_i \cdot L_i \cdot x_{i,0}.$$

The value of the portfolio tomorrow is then

$$V_1 = \sum_{i=1}^{n} \omega_i \cdot L_i \cdot x_{i,1},$$

where $x_{i,1}$ is the exchange rate tomorrow from the ith base currency to the quote currency. Similarly, the value of V_1 is unknown today since the exchange rates tomorrow are unknown.

Thus, the change in the value of the portfolio is

$$\begin{aligned}
\Delta V &= V_1 - V_0 \\
&= \sum_{i=1}^{n} \omega_i \cdot L_i \cdot (x_{i,1} - x_{i,0}) \\
&= \sum_{i=1}^{n} \omega_i \cdot L_i \cdot x_{i,0} \cdot \frac{x_{i,1} - x_{i,0}}{x_{i,0}} \\
&\approx \sum_{i=1}^{n} \delta_i \cdot R_i,
\end{aligned}$$

where $\delta_i = \omega_i \cdot L_i \cdot x_{i,0}$ and $R_i = \ln\left(\dfrac{x_{i,1}}{x_{i,0}}\right)$.

Let $y_{i,j}$ be the exchange rate from the ith base currency to the quote currency j days ago where $j = 0, 1, \ldots, N$. R_i is assumed to be normally distributed $R_i \sim N(0, \sigma_i^2)$, where σ_i is the standard deviation of the historical returns $\{R_{i,j}\}$, where $R_{i,j} = \ln\left(\frac{y_{i,j}}{y_{i,j+1}}\right)$. Additionally, let $\rho_{i,i'}$ be the correlation between the sets $\{R_{i,j}\}$ and $\{R_{i',j}\}$. Since ΔV is the sum of normally distributed random variables, then it is also normally distributed with mean 0 and standard deviation given by

$$\begin{aligned}\sigma &= \sqrt{\boldsymbol{W}\boldsymbol{S}\boldsymbol{W}^{\boldsymbol{T}}}\\ &= \sqrt{\sum_{i=1}^{n}(\delta_i \cdot \sigma_i)^2 + 2\sum_{i<i'}\delta_i \cdot \delta_{i'} \cdot \sigma_i \cdot \sigma_{i'} \cdot \rho_{i,i'}},\end{aligned}$$

where $\boldsymbol{W} = [\delta_1, \delta_2, \ldots, \delta_n]$ and $\boldsymbol{S} = [\mathrm{Cov}_{i,i'}]$ is the covariance matrix.

Hence, the 1-day 99% VaR, denoted by $|X|$, is given by

$$\begin{aligned}\mathbb{P}(\Delta V \geq X) &= 0.99\\ \mathbb{P}\left(\frac{\Delta V}{\sigma} \geq \frac{X}{\sigma}\right) &= 0.99\\ \mathbb{P}\left(\frac{\Delta V}{\sigma} \leq \frac{X}{\sigma}\right) &= 0.01\\ F\left(\frac{X}{\sigma}\right) &= 0.01\\ X &= \sigma \cdot F^{-1}(0.01)\\ |X| &= \sigma \cdot F^{-1}(0.99),\end{aligned}$$

where $F(\cdot)$ is the cumulative distribution function (cdf) of the standard normal distribution.

2.1.3 Exponentially Weighted Moving Average Model (EWMA) Model for Volatility

As an alternative, the standard deviation and correlations under the delta-normal approach can be calculated by the exponentially weighted moving average (EWMA) model. For a random variable X with sample values $x_1, x_2, \ldots, x_N$ and mean 0, the standard deviation σ of the sample is given by

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{j=1}^{N} x_j^2}.$$

Replacing $N-1$ in the denominator by N, $\sigma = \sqrt{\frac{1}{N}\sum_{j=1}^{N} x_j^2}$. This says that the standard deviation estimates the volatility of X by giving equal weights to all x_j^2.

Suppose that $x_1, x_2, \ldots, x_N$ are assigned subscripts according to time with x_1 representing the most recent data. Thus, if X represents return, then x_j corresponds to the return using the rates j days and $j+1$ days ago. In the EWMA model, the volatility is estimated by giving higher weights to more recent data. Hence, the volatility υ is given by

$$\upsilon = \sqrt{\sum_{j=1}^{N} \alpha_j \cdot x_j^2},$$

where α_j is positive and corresponds to the weight given to x_j^2, $\alpha_j > \alpha_{j'}$ when $j > j'$, and the sum of all the weights is $\sum_{j=1}^{N} \alpha_j = 1$. The weights α_j for $j = 1, 2, \ldots, N$, are given by

$$\alpha_j = (1-\lambda)\,\lambda^{j-1},$$

where $\lambda \in (0,1)$ is the decay factor. Note that $\sum_{j=1}^{N} \alpha_j = 1 - \lambda^N$, and when N is large, $\lambda^N \approx 0$. The decay factor is also determined based on the chosen half-life, which is the exponent t such that $\lambda^t = \frac{1}{2}$.

Furthermore, suppose that there are two random variables X and Y with sample values of x_j and y_j, respectively, indexed by time where $j = 1, 2, \ldots, N$. If X and Y both have mean 0, then the covariance of X and Y is given by

$$\operatorname{Cov}(X,Y) = \frac{1}{N}\sum_{j=1}^{N} x_j \cdot y_j.$$

Applying the EWMA model where the equal weights can be replaced with decreasing weights, the covariance of X and Y can be written as

$$\operatorname{Cov}_{\text{EWMA}}(X,Y) = \sum_{j=1}^{N} (1-\lambda)\lambda^{j-1} \cdot x_j \cdot y_j.$$

To illustrate, consider the half-life of 26 weeks (half year), which is equivalent to 130 days assuming 260 trading days or 52 weeks in a year. Then, $\lambda^{130} = 0.5$ so that

$$\lambda = \exp\left(\frac{\ln(0.5)}{130}\right) \approx 0.994682.$$

Thus, the volatility υ can be estimated by

$$\upsilon = \sqrt{\sum_{j=1}^{N} (1-0.994682)(0.994682)^{j-1} \cdot x_j^2}.$$

Moreover, the covariance of X and Y by the EWMA model is given by

$$\operatorname{Cov}_{\text{EWMA}}(X,Y) = \sum_{j=1}^{N} (1-0.994682)\ (0.994682)^{j-1} \cdot x_j \cdot y_j.$$

The volatility υ, covariance $\operatorname{Cov}_{\text{EWMA}}$, and λ can be used to compute for the VaR of a portfolio.

2.2 Assumptions

The following assumptions are made in the calculation of the 1-day 99% undiversified and diversified VaR for an FX spot portfolio:

1. For consistency, the portfolio consists of four base currencies which are Japanese Yen (JPY), Euro (EUR), Australian Dollar (AUD), and Philippine Peso (PHP).
2. VaR is calculated using the quote currency of US Dollar (USD). Afterwards, the VaR obtained is converted to Philippine Peso (PHP) using the exchange rate of 49.73 PHP/USD, the exchange rate as of position date.
3. The returns in the exchange rate R_i are assumed to be normally distributed with mean 0 and variance σ_i^2.
4. The position date is November 29, 2016 and 260 logarithmic returns on FX rates are considered. Thus, the time period considered is from November 9, 2015 to November 29, 2016.
5. The EWMA model for volatility is considered in computing for the standard deviation and covariance of the portfolio. Lambda λ is assumed to be 0.994682.

2.3 Excel Implementation

The calculation of the 1-day 99% VaR using Excel is shown below.

1. First, the FX positions and Implied FX rates (to PHP and to USD) are determined. For consistency, the USD position is converted into PHP position since USD is considered the quote currency for the entire portfolio. Hence, PHP is considered a base currency.

Base Currency	Position in Base	Position in PHP	Implied FX rate (Base to PHP)	Position in USD	Implied FX rate (Base to USD)
AUD	69,737.15	2,592,477.00	37.17498	51,472.99	0.7381
EUR	161,600.30	8,544,656.00	52.87525	171,296.30	1.0600
JPY	3,012,807.00	1,329,090.00	0.44115	26,340.97	0.0087
USD	(1,805,769.70)	(89,889,409.00)	49.77900	(1,805,769.73)	1.0000
PHP	(89,889,409.00)	(89,889,409.00)	1.00000	(1,805,769.73)	0.0201
Total		**(77,423,186.000)**		**(1,556,659.47)**	

Figure 1: FX Spot: FX Positions and Implied FX Rates

2. Next, the FX historical data is converted in terms of the quote currency USD. Specifically, the JPY and PHP data are converted by taking their inverse to represent a quote currency of USD.
3. Afterward, for each of the base currencies, 260 logarithmic returns are computed following

$$R_{i,j} = \ln\left(\frac{q_{i,j}}{q_{i,j+1}}\right),$$

where $q_{i,j}$ is the exchange rate for the ith base currency j days ago.

Scenario	Date	Weight	JPY/USD	Log Return	EUR/USD	Log Return	AUD/USD	Log Return	PHP/USD	Log Return
1	11/29/2016	0.0053	0.0089	-0.0028	1.0627	0.0040	0.7472	-0.0001	0.0201	-0.0004
2	11/28/2016	0.0053	0.0089	0.0077	1.0585	-0.0004	0.7473	0.0040	0.0201	0.0026
3	11/25/2016	0.0053	0.0088	0.0014	1.0589	0.0032	0.7443	0.0054	0.0201	0.0028
4	11/24/2016	0.0052	0.0088	-0.0074	1.0555	-0.0002	0.7403	0.0022	0.0200	-0.0024
5	11/23/2016	0.0052	0.0089	-0.0134	1.0557	-0.0064	0.7387	-0.0009	0.0201	-0.0002
6	11/22/2016	0.0052	0.0090	0.0005	1.0625	0.0018	0.7394	0.0050	0.0201	-0.0004
7	11/21/2016	0.0052	0.0090	-0.0017	1.0606	0.0017	0.7357	0.0026	0.0201	-0.0010
8	11/18/2016	0.0051	0.0090	-0.0094	1.0588	-0.0045	0.7338	-0.0119	0.0201	-0.0044
9	11/17/2016	0.0051	0.0091	-0.0061	1.0636	-0.0056	0.7426	-0.0082	0.0202	-0.0042

Figure 2: FX Spot: Logarithmic Returns and EWMA Weights

At the same time, since the EWMA model is used for the calculation of the standard deviation, the weights for each scenario are also calculated.

4. Using the logarithmic return and weights computed, the standard deviation for the ith base currencies υ_i is calculated. This is given by

$$\upsilon_i = \sqrt{\sum_{j=1}^{N} \alpha_j \cdot R_{i,j}^2}.$$

This is calculated in Excel using the `SUMPRODUCT()` function. The volatilities obtained are 0.00671 (JPY), 0.00438 (EUR), 0.00601 (AUD), and 0.00245 (PHP).

5. Next, the VaR for the ith base currency is obtained as

$$\text{VaR}_i = L_i \cdot q_{i,0} \cdot \upsilon_i \cdot F^{-1}(0.99),$$

where L_i is the position in the ith base currency and $q_{i,0}$ is the exchange rate today. The undiversified VaR is then calculated by taking the sum of the VaRs.

$$\text{VaR}_{\text{Portfolio}} = \sum_{i=1}^{n} \text{VaR}_i.$$

Currency	Position	FX Rate	Stdev	1-day 99% VaR (USD)	USD/PHP Rate	1-day 99% VaR (PHP)
JPY	3,012,807.00	0.0087	0.00671	411.31	49.73	20,454.40
EUR	161,600.30	1.0600	0.00438	1,745.16	49.73	86,786.70
AUD	69,737.15	0.7381	0.00601	719.99	49.73	35,805.19
PHP	(89,889,409.00)	0.0201	0.00245	10,280.13	49.73	511,230.62
1-day 99% VaR (USD)	13,156.58	*Undiversified*				
1-day 99% VaR (PHP)	654,276.92	*Undiversified*				

Figure 3: FX Spot: Undiversified VaR

The 1-day 99% Undiversified VaR is then 13,156.58 USD or 654,276.92 PHP, where the exchange rate used is 49.73 PHP/USD.

6. Afterward, the covariance matrix is calculated using the EWMA model to produce the covariance between each pair of base currencies.

$$\text{Cov}_{\text{EWMA}}(Q_i, Q_{i'}) = \sum_{j=1}^{260} \alpha_j \cdot R_{i,j} \cdot R_{i',j}.$$

Moreover, the position vector $\boldsymbol{W} = [\delta_1, \delta_2, \ldots, \delta_n]$ is also obtained where

$$\delta_i = \omega_i \cdot L_i \cdot q_{i,0}.$$

	JPY	EUR	AUD	PHP
JPY	0.0000451	0.0000117	0.0000046	0.0000010
EUR	0.0000117	0.0000192	0.0000117	0.0000029
AUD	0.0000046	0.0000117	0.0000362	0.0000049
PHP	0.0000010	0.0000029	0.0000049	0.0000060

JPY	EUR	AUD	PHP
26,340.97	171,296.30	51,472.99	(1,805,769.70)

Figure 4: FX Spot: Covariance Matrix $\boldsymbol{S}$ and Position Vector $\boldsymbol{W}$

7. Finally, the 1-day 99% Diversified VaR is calculated as

$$\text{VaR} = \sigma \cdot F^{-1}(0.99),$$

where $\sigma = \sqrt{\boldsymbol{W}\boldsymbol{S}\boldsymbol{W}^{\boldsymbol{T}}}$. The portfolio volatility is obtained to be 4,212.03 (in USD). Hence, the diversified VaR is 9,798.64 USD or 487,286.15 PHP, where the exchange rate used is 49.73 PHP/USD.

1-day 99% VaR (USD)	9,798.64	*Diversified*
1-day 99% VaR (PHP)	487,286.15	*Diversified*

Figure 5: FX Spot: Diversified VaR

2.4 Results and Discussion

To summarize, the 1-day 99% undiversified VaR is 654,276.92 PHP while the 1-day 99% diversified VaR is 487,286.15 PHP. As expected, the diversified VaR is lower than the undiversified VaR since the former considers the correlations between the risk factors of the assets in the portfolio. Hence, the diversified VaR has lower measured potential losses for the portfolio.

3 VaR for Peso Fixed Income Portfolio

3.1 Conceptual Discussion

3.1.1 Single-Asset Portfolio

Consider a fixed-rate bond with principal P, annual coupon rate c compounded m times per year, yield to maturity y compounded m times per year, and N years to maturity. The price of the bond at issue date or any coupon date is

$$\begin{aligned} B &= \frac{\frac{c}{m}P}{\left(1+\frac{y}{m}\right)} + \frac{\frac{c}{m}P}{\left(1+\frac{y}{m}\right)^2} + \cdots + \frac{\frac{c}{m}P}{\left(1+\frac{y}{m}\right)^{mN}} + \frac{P}{\left(1+\frac{y}{m}\right)^{mN}} \\ &= \sum_{i=1}^{mN} \frac{\frac{c}{m}P}{\left(1+\frac{y}{m}\right)^i} + \frac{P}{\left(1+\frac{y}{m}\right)^{mN}}. \end{aligned}$$

However, if the bond is priced between coupon dates, then the accrued interest for the time period between the settlement date and the previous coupon date must be taken into account. This amount must be paid to the previous owner of the bond for holding the bond after the previous coupon date. The accrued interest is given by

$$\text{Accrued Interest} = P \cdot \frac{c}{m} \cdot \frac{t}{T},$$

where t is the number of days from the previous coupon date to the settlement date and T is number of days between the coupon dates.

Therefore, the clean price of a bond is given by

$$\text{Clean Bond Price} = \text{Dirty Bond Price} - \text{Accrued Interest}.$$

In calculating the VaR of a fixed income asset, the dirty price is considered. The dirty price of the bond today is given by

$$B_0 = B(y_0),$$

where y_0 is the yield to maturity today.

The dirty price of the bond tomorrow is then

$$B_1 = B(y_1),$$

where y_1 is the yield to maturity tomorrow. B_1 is unknown today since y_1 is unknown.

Thus, the change in the price of the bond is

$$\Delta B = B_1 - B_0 = B(y_1) - B(y_0).$$

Alternatively, the change in bond price ΔB can also be approximated using the modified duration D^*.

First, the Macaulay duration D is defined as the weighted average time until receipt of the bond's cash flows, where the weights are the present value of the cash flows. Mathematically,

$$D = \sum_{i=1}^{n} t_i \frac{PV(c_i)}{B} = t_1 \cdot \frac{PV(c_1)}{B} + t_2 \cdot \frac{PV(c_2)}{B} + \cdots + t_n \cdot \frac{PV(c_n)}{B},$$

where c_i is the ith coupon payment on payment date t_i, c_n is the last coupon payment plus the bond's principal, $PV(\cdot)$ is the present value function, and B is the bond price. Thus, $\dfrac{PV(c_i)}{B}$ represents the weight given to time t_i.

In order to derive the formula for the modified duration D^*, consider the following. Suppose the yield to maturity is y compounded m times a year and the coupon is c_i paid

m times per year, where c_n includes the principal payment. Then, the bond price is given by

$$B = \sum_{i=1}^{mN} \frac{c_i}{\left(1 + \frac{y}{m}\right)^i}.$$

The corresponding Macaulay duration is given by

$$D = \sum_{i=1}^{mN} \frac{i}{m} \cdot \frac{\frac{c_i}{\left(1 + \frac{y}{m}\right)^i}}{B}.$$

Suppose that B is a differential function of the yield to maturity y, then the derivative of B with respect to y is given by

$$\begin{aligned}
\frac{dB}{dy} &= \sum_{i=1}^{mN} -\frac{i}{m} \cdot \frac{c_i}{\left(1 + \frac{y}{m}\right)^{i+1}} \\
&= -\frac{1}{\left(1 + \frac{y}{m}\right)} \cdot \sum_{i=1}^{mN} \left(\frac{i}{m}\right) \cdot \frac{c_i}{\left(1 + \frac{y}{m}\right)^i} \\
&= -\frac{B}{\left(1 + \frac{y}{m}\right)} \cdot \sum_{i=1}^{mN} \left(\frac{i}{m}\right) \cdot \frac{\frac{c_i}{\left(1 + \frac{y}{m}\right)^i}}{B} \\
&= -\frac{B}{\left(1 + \frac{y}{m}\right)} \cdot D.
\end{aligned}$$

Let $D^* = \dfrac{D}{\left(1 + \frac{y}{m}\right)}$ be the modified duration. Therefore,

$$\frac{dB}{dy} = -BD^*.$$

The modified duration D^* can be interpreted as the percentage change in the bond price given a change in the yield. From Calculus, the partial derivative $\dfrac{dB}{dy}$ can be approximated as $\dfrac{\Delta B}{\Delta y}$ when Δy is sufficiently small. Thus, using the modified duration, the change in the bond price ΔB can be approximated as

$$\Delta B \approx -BD^* \Delta y.$$

Using the historical simulation approach, the 1-day 99% VaR is obtained by considering the changes in the bond price given the historical 1-day changes in interest rates. Let y_j be the relevant interest rates j days ago, where $j = 0, 1, \ldots, N$. Then, the one-day change in interest rate j days ago is $\Delta y_j = y_j - y_{j+1}$. It is important to note that a positive change in interest rate Δy_j implies a negative change in the bond price since higher interest rates leads to lower bond prices.

Finally, the approximate change in the bond price for the jth scenario is given by $-BD^*\Delta y_j$. Taking the absolute value of the first percentile of these approximations gives the 1-day 99% VaR under the historical simulation approach.

3.1.2 Multi-Asset Portfolio

The Value-at-Risk of a portfolio with multiple fixed income assets can be calculated in two ways: the undiversified VaR and the diversified VaR. The undiversified VaR is calculated by simply adding the individual VaR amounts for each asset. Mathematically,

$$\text{VaR}_{\text{Portfolio}} = \sum_{i=1}^{n} \text{VaR}_i,$$

where VaR_i is the VaR of the ith asset in the portfolio. To reiterate, this method does not take into account the correlation of the risk factors of the different assets in the portfolio.

To calculate the diversified VaR, consider a portfolio of n fixed income assets, each with yield to maturity $y_{i,0}$ and remaining time to maturity T_i, where $i = 1, 2, \ldots, n$. The value of the portfolio today is the sum of the dirty prices of each bond, denoted by $B_{i,0}$.

$$V_0 = \sum_{i=1}^{n} B\left(y_{i,0}\right).$$

The value of the portfolio tomorrow is then

$$V_1 = \sum_{i=1}^{n} B\left(y_{i,1}\right),$$

with $y_{i,1}$ as the yield to maturity tomorrow. The value of V_1 is unknown since the values of $y_{i,1}$ are unknown today.

Thus, the change in the value of the portfolio is

$$\begin{aligned}
\Delta V &= V_1 - V_0 \\
&= \sum_{i=1}^{n} B\left(y_{i,1}\right) - B\left(y_{i,0}\right) \\
&= \sum_{i=1}^{n} \Delta B_i \\
&\approx \sum_{i=1}^{n} -B_i D_i^* \Delta y_i,
\end{aligned}$$

where B_i is the price of the ith bond today, D_i^* is the modified duration of the ith bond, and Δy_i is the change in the yield of the ith bond.

In computing the diversified VaR using the historical simulation approach, it is assumed that each of the Δy_i's are uniformly distributed with values $\{\Delta y_{i,j}\}$, where $\Delta y_{i,j} = y_{i,j} - y_{i,j+1}$. Then, for each scenario j, the change in the portfolio value is

$$\Delta V_j = \sum_{i=1}^{n} -B_i D_i^* \Delta y_{i,j}.$$

Finally, the 1-day 99% diversified VaR is obtained by taking the absolute value of the first percentile of these approximate changes in the portfolio value.

3.2 Assumptions

The following assumptions are made in the calculation of the 1-day 99% undiversified and diversified VaR for a fixed income portfolio:

1. The portfolio consists of two bonds: PIBL1218H283 and PIID0320D087. The first is a zero-coupon bond while the second is a 3-year coupon-bearing bond with semi-annual coupons. The day count conventions used are Actual/360 and European 30/360, respectively.

2. The position date for the calculation of the VaR is November 6, 2018.

3. The VaR is obtained using the historical simulation approach, the modified duration approach, and 260 scenarios.

4. The actual daily change of interest rates is used. The yields are obtained by interpolating between PDST-R2 / PHP BVAL benchmark rates nearest to the remaining maturity of the bond.

5. The 1st percentile of the change in the value of the portfolio is calculated using the `PERCENTILE()` function in Excel.

3.3 Excel Implementation

The calculation of the 1-day 99% undiversified and diversified VaR using Excel is shown below.

1. First, the relevant information of each bond are obtained and summarized. These include the dirty bond price, modified duration, and remaining maturity.

	Dirty Price (B_0)	Modified Duration (D^*)	Maturity
Bond 1	26,528,809.83	0.77917	0.81944
Bond 2	195,043.03	1.35630	1.43056

Figure 6: Fixed Income: Relevant Bond Information

2. From the given PDST R2 / PHP BVAL rates, the yields for the bonds are calculated by interpolation using the `FORECAST()` function. Since the first bond has a maturity of 0.8194 years, the 6-month and 1-year rates are considered. Since the second bond has a maturity of 1.4306 years, the 1-year and 2-year rates are considered.

3. Using the interpolated interest rates, the 1-day change in interest rates $\Delta y_{i,j} = y_{i,j} - y_{i,j+1}$ are generated, where $y_{i,j}$ is the interest rate j days ago applicable to bond i. 260 scenarios are considered.

Interpolation for PIBL1218H283					
Scenario	Date	Tenor 0.50	1.00	0.82	Change
1	11/6/2018	6.01	6.58	6.37	0.0168%
2	11/5/2018	5.97	6.57	6.36	0.0277%
3	10/31/2018	5.92	6.56	6.33	0.0258%
4	10/30/2018	5.86	6.55	6.30	0.0244%
5	10/29/2018	5.84	6.53	6.28	0.0410%
6	10/26/2018	6.14	6.30	6.24	-0.0019%
7	10/25/2018	5.92	6.42	6.24	-0.0271%
8	10/24/2018	5.95	6.45	6.27	0.0191%

Interpolation for PIID0320D087					
Scenario	Date	Tenor 1.00	2.00	1.43	Change
1	11/6/2018	6.58	6.86	6.70	0.0043%
2	11/5/2018	6.57	6.86	6.70	0.0233%
3	10/31/2018	6.56	6.82	6.67	-0.0071%
4	10/30/2018	6.55	6.85	6.68	-0.0070%
5	10/29/2018	6.53	6.90	6.69	0.1834%
6	10/26/2018	6.30	6.78	6.50	-0.1272%
7	10/25/2018	6.42	6.91	6.63	-0.0448%
8	10/24/2018	6.45	6.98	6.68	-0.3355%

Figure 7: Fixed Income: Interpolation of Interest Rates

4. For the jth scenario, the 1-day change in the price of ith bond is calculated using the modified duration method, $\Delta V_{i,j} = -B_i D_i^* \Delta y_{i,j}$, where B_i is the dirty price of bond i today and D_i^* is the modified duration of bond i. Lastly, for each scenario j, the sum of the 1-day change in bond price across all assets is computed. This is the change in the total value of the portfolio for the jth scenario.

Scenario	Change in Bond 1	Change in Bond 2	Total Change in Portfolio
1	(3,479.53)	(11.32)	(3,490.85)
2	(5,724.58)	(61.76)	(5,786.34)
3	(5,334.14)	18.74	(5,315.40)
4	(5,035.56)	18.52	(5,017.04)
5	(8,472.03)	(485.19)	(8,957.22)
6	401.93	336.36	738.29
7	5,597.11	118.42	5,715.53
8	(3,948.06)	887.61	(3,060.46)
9	(21,537.51)	(1,165.43)	(22,702.94)

Figure 8: Fixed Income: Change in the Value of the Portfolio

5. Finally, the 1-day 99% diversified VaR is obtained by getting the absolute value of the first percentile of these approximations. The first percentile is obtained using the `PERCENTILE()` function in Excel.

1-day 99% Diversified VaR	142,159.21
1-day 99% Undiversified VaR	142,387.39

Figure 9: Fixed Income: Undiversified and Diversified VaR

3.4 Results and Discussion

In summary, the 1-day 99% undiversified VaR is 142,387.39 PHP while the 1-day 99% diversified VaR is 142,159.21 PHP. The undiversified VaR is obtained by simply adding the absolute value of the first percentile of each bond considered. As expected, the diversified VaR is less than the undiversifed VaR due to the benefits of diversification. However, it is worth noting that the difference is not significant.

4 VaR and TVaR for FX Forward

4.1 Conceptual Discussion

Consider the base currency C_b, the quote currency C_q, and S_t the exchange rate at time t from C_b to C_q, *i.e.*, one unit of C_b is worth S_t units of C_q. Additionally, let $r_{b,t}$ be the simple zero rate (money market interest rate) in currency C_b with tenor $T-t$, and $r_{q,t}$ be the simple zero rate (money market interest rate) in currency C_q with tenor $T-t$. In a foreign exchange forward, the long position party agrees to buy the base currency, in terms of the quote currency, for the forward exchange rate F_0 at the delivery date $t=T$, where F_0 is determined at deal date $t=0$. The forward exchange rate F_t at time t from the base currency C_b to the quote currency C_q is given by

$$F_t = S_t \cdot \frac{1 + r_{q,t}\,(T-t)}{1 + r_{b,t}\,(T-t)}.$$

This equation is also referred to as the interest rate parity which relates exchange rates and interest rates.

The value of the FX forward contract at time t to the party with long position is

$$V_t = \frac{F_t - K}{1 + r_{q,t}\,(T-t)} \cdot L_b,$$

where K is the contract forward rate fixed at time $t=0$, L_b is the principal amount in the base currency, and $r_{q,t}$ is the interest rate in currency C_q at time t. A positive value for V_t indicates a gain for the party with long position, while a negative value indicates a loss for the party with long position.

The value of the FX forward contract at time $t+1$ to party with the long position is then

$$V_{t+1} = \frac{F_{t+1} - K}{1 + r_{q,t+1}\,(T-(t+1))} \cdot L_b,$$

Thus, the change in the value of the FX forward contract is

$$\Delta V = V_{t+1} - V_t.$$

Similar to the previous sections, the 1-day 99% VaR of an FX forward contract, denoted as $|X|$, is the value X such that

$$\mathbb{P}\,(\Delta V \geq X) = 0.99.$$

In addition, the Tail Value-at-Risk (TVaR) at $(100c)\%$ confidence level is the expected loss given that the loss exceeds the $(100c)\%$ VaR. Suppose $|X|$ is the VaR, then the TVaR is computed as

$$\text{TVaR}_c = \mathbb{E}\left[\Delta V | \Delta V < X\right].$$

For consistency, the VaR and TVaR are reported as positive quantities.

4.1.1 Historical Simulation Approach

Using the historical simulation approach, the 1-day 99% VaR is calculated using the historical values of the forward rates $\{F_{t+1,j}\}$ and money market rates $\{r_{q,t+1,j}\}$, where $j = 1, 2, \ldots, N$. The possible values for V_{t+1} are first generated following

$$V_{t+1,j} = \frac{F_{t+1,j} - K}{1 + r_{q,t+1,j}\left(T - (t+1)\right)} \cdot L_b.$$

Afterward, the one-day change in the value of the FX forward contract is calculated using $\Delta V_j = V_{t+1,j} - V_{t,0}$, where $V_{t,0}$ is the value of the FX forward contract today.

The 1-day 99% VaR is then taken as the absolute value of the first percentile of these one-day changes. Moreover, the 1-day 99% TVaR is calculated by getting the absolute value of the mean of the observations smaller than the VaR.

4.1.2 BRW Approach

The BRW approach, based on the work of J. Boudoukh, M.P. Richardson, and R. Whitelaw, is a modification of the historical simulation approach for calculating VaR. First, scenarios for the one-day change in the value of the FX forward contract are obtained the same way as the historical simulation approach. However, more recent historical data are given higher weights rather than having equal weights.

Suppose the change in the value of the FX forward contract under scenario j is denoted by $\Delta V_j = V_{t+1,j} - V_{t,0}$, where $V_{t+1,j}$ is the value of the contract tomorrow based on the market rates j days ago and $V_{t,0}$ is the value of the contract today. The most recent change in the value of the contract ΔV_1 is assigned the highest probability

$$p = \frac{1 - \lambda}{1 - \lambda^N},$$

where $\lambda \in (0, 1)$ is the decay factor. In general, ΔV_j is assigned the probability of $p\lambda^{j-1}$, where this probability decreases as j increases since $\lambda < 1$, giving higher weights for the more recent historical data.

Moreover, using the formula for the sum of a finite geometric series,

$$1 + \lambda + \lambda^2 + \cdots + \lambda^{N-1} = \frac{1 - \lambda^N}{1 - \lambda},$$

the sum of the weights $p\lambda^{j-1}$ is equal to 1.

$$\begin{aligned}\sum_{j=1}^{N} p\lambda^{j-1} &= p\left(1+\lambda+\lambda^2+...+\lambda^{N-1}\right)\\ &= p\left(\frac{1-\lambda^N}{1-\lambda}\right)\\ &= \left(\frac{1-\lambda}{1-\lambda^N}\right)\left(\frac{1-\lambda^N}{1-\lambda}\right)\\ &= 1.\end{aligned}$$

In calculating the 1-day 99% VaR, suppose the resulting (increasing) order of the values of ΔV_j is

$$\{\Delta V_{j_1}, \Delta V_{j_2}, \cdots, V_{j_N}\},$$

where $\Delta V_{j_k} < \Delta V_{j_{k'}}$ when $k < k'$. Then, the cumulative distribution function $F(\cdot)$ is defined as

$$F(\Delta V_{j_m}) = \sum_{k=1}^{m} p\lambda^{j_k - 1}.$$

Suppose $\Delta V_{j_{n+1}}$ is the largest value such that

$$1 - F\left(\Delta V_{j_{n+1}}\right) < 0.99 < 1 - F\left(\Delta V_{j_n}\right).$$

Then, the 1-day 99% VaR using the BRW approach is obtained by interpolating between ΔV_{j_n} and $\Delta V_{j_{n+1}}$, where the x-coordinates are $1 - F\left(\Delta V_{j_n}\right)$ and $1 - F\left(\Delta V_{j_{n+1}}\right)$, respectively. Moreover, the 1-day 99% TVaR is calculated by getting the absolute value of the weighted average of the observations smaller than the VaR, where the weights are proportional to the corresponding BRW probabilities.

4.2 Assumptions

The following assumptions are made in the calculation of the 1-day 99% VaR for an FX forward contract:

1. The base currency is EUR while the quote currency is USD. The deal date of the contract was April 25, 2016 and its delivery date is October 25, 2016. At deal date, the parties agreed to exchange 1 million EUR for 1,138,500 USD on the delivery date.

2. The date today is June 30, 2016 and the USD/PHP rate is 47.06.

3. An Actual/360 day count convention is used throughout the calculations. The relevant interest rates are interpolated from the US LIBOR and EURIBOR benchmark rates using the `FORECAST()` function.

4. The VaR and TVaR calculations are for the party buying the base currency and 260 scenarios are considered. The VaR and TVaR are calculated in USD and later converted to PHP. The BRW approach uses a lambda of $\lambda = 99.1\%$

4.3 Excel Implementation

The calculation of the 1-day 99% VaR using Excel is shown below.

1. First, using information on the FX forward contract, the contract forward rate K is determined.

Contract Forward Rate	1.1385
USD	1,138,500
EUR	1,000,000

Figure 10: FX Forward: Contract Information

2. Afterward, the relevant dates are identified and their corresponding number of days from today and tomorrow are calculated.

	Deal Date	Date Today	Date Tomorrow	Delivery Date	3M	6M
Date	4/25/2016	6/30/2016	7/1/2016	10/25/2016	9/30/2016	12/30/2016
Days from today			1	117	92	183
Days remaining		117	116			

Figure 11: FX Forward: Relevant Dates

3. Next, the relevant US LIBOR and EURIBOR rates and tenors are identified and the forward rate today is calculated. The market value of the contract today is also calculated.

US LIBOR		
Tenor	Days	Rate
3 months	92	0.61845%
6 months	183	0.92500%
117 days	117	0.70267%
Forward Rate		
Tenor	Days	Rate
3 months	92	1.1132
6 months	183	1.1170
117 days	117	1.1143
Value of FX Forward today		(24,183.33)

Figure 12: FX Forward: Market Value Today

4. Then, the change in the market value of the contract tomorrow is calculated. First, using the 3M and 6M US LIBOR and EURIBOR rates and the FX spot rates today, the 3M and 6M US LIBOR and EURIBOR rates and the FX spot rates tomorrow are obtained. Using the interest rate parity, the 3M and 6M forward rates obtained.

Scenario	Date	US LIBOR						EURIBOR						Spot			FX Forward	
		3M	Change	3M tomorrow	6M	Change	6M tomorrow	3M	Change	3M tomorrow	6M	Change	6M tomorrow	Today	Change	Tomorrow	3M tomorrow	6M tomorrow
1	6/30/2016	0.62%	0.01%	0.63%	0.93%	-0.27%	0.66%	-0.27%	0.00%	-0.27%	-0.19%	-0.01%	-0.20%	1.1107	-0.0018	1.1087	1.1113	1.1136
2	6/29/2016	0.61%	0.00%	0.61%	1.19%	0.28%	1.21%	-0.27%	0.00%	-0.27%	-0.18%	0.00%	-0.19%	1.1127	0.0053	1.1186	1.1211	1.1266
3	6/28/2016	0.61%	0.00%	0.62%	0.91%	-0.09%	0.84%	-0.27%	-0.02%	-0.29%	-0.18%	0.01%	-0.18%	1.1068	0.0034	1.1106	1.1132	1.1164
4	6/27/2016	0.61%	-0.02%	0.59%	1.00%	-0.15%	0.78%	-0.25%	0.00%	-0.27%	-0.19%	0.00%	-0.20%	1.1030	0.0011	1.1042	1.1066	1.1097
5	6/24/2016	0.63%	0.01%	0.63%	1.14%	0.03%	0.96%	-0.25%	0.01%	-0.26%	-0.19%	-0.03%	-0.22%	1.1018	-0.0334	1.0650	1.0674	1.0714
6	6/23/2016	0.62%	-0.02%	0.60%	1.11%	0.17%	1.09%	-0.26%	-0.01%	-0.28%	-0.16%	0.00%	-0.19%	1.1392	0.0082	1.1485	1.1511	1.1560
7	6/22/2016	0.64%	0.01%	0.63%	0.95%	0.01%	0.93%	-0.25%	0.00%	-0.27%	-0.16%	-0.01%	-0.20%	1.1299	0.0049	1.1354	1.1380	1.1419
8	6/21/2016	0.63%	-0.04%	0.58%	0.94%	0.00%	0.93%	-0.25%	0.00%	-0.27%	-0.16%	0.00%	-0.20%	1.1244	-0.0066	1.1170	1.1195	1.1234
9	6/20/2016	0.67%	-0.01%	0.61%	0.94%	-0.16%	0.77%	-0.25%	0.02%	-0.25%	-0.15%	0.02%	-0.17%	1.1318	-0.0015	1.1301	1.1326	1.1355

Figure 13: FX Forward: Interest Rates and Forward Rates Changes

Afterward, using the generated data, the 116-day forward rates and 116-day US LIBOR rates are by interpolating between the 3M and 6M rates using the `FORECAST()` function. Finally, the market value of the contract tomorrow is obtained as well as the change in the market value tomorrow.

Scenario	US LIBOR			Forward Rates			MV tomorrow	ΔMV
	3M	6M	116 days	3M	6M	116 days		
1	0.63%	0.66%	0.64%	1.1113	1.1136	1.1119	(26,582.82)	(2,399.49)
2	0.61%	1.21%	0.77%	1.1211	1.1266	1.1226	(15,889.89)	8,293.44
3	0.62%	0.84%	0.68%	1.1132	1.1164	1.1140	(24,414.09)	(230.76)
4	0.59%	0.78%	0.64%	1.1066	1.1097	1.1074	(30,993.39)	(6,810.07)
5	0.63%	0.96%	0.72%	1.0674	1.0714	1.0685	(69,859.46)	(45,676.13)
6	0.60%	1.09%	0.73%	1.1511	1.1560	1.1524	13,887.11	38,070.44
7	0.63%	0.93%	0.71%	1.1380	1.1419	1.1390	539.84	24,723.17
8	0.58%	0.93%	0.67%	1.1195	1.1234	1.1205	(17,966.62)	6,216.71
9	0.61%	0.77%	0.65%	1.1326	1.1355	1.1333	(5,149.68)	19,033.65
10	0.64%	1.08%	0.75%	1.1472	1.1520	1.1485	9,948.04	34,131.37

Figure 14: FX Forward: Market Value Tomorrow

5. For the historical simulation approach, the first, second, and third smallest values of ΔV_j are determined using the `SMALL()` function. Then, the 1-day 99% VaR in USD is computed by interpolating between the second and third smallest value of ΔV_j. The 1-day 99% TVaR in USD is obtained by taking the average of the first and second smallest values of ΔV_j. Lastly, the VaR and TVaR are also reported in PHP using the rate 47.06 USD/PHP.

Historical Simulation			
1st Smallest	(57,470.76)		1
2nd Smallest	(56,662.76)		2
3rd Smallest	(50,953.52)		3
1-day 99% VaR	53,237.21	*USD*	
1-day 99% TVaR	57,066.76	*USD*	
1-day 99% VaR	2,505,343.32	*PHP*	
1-day 99% TVaR	2,685,561.56	*PHP*	

Figure 15: FX Forward: VaR and TVaR (Historical Simulation Approach)

6. Finally, the VaR and TVaR is also calculated using the BRW approach. With $\lambda = 99.1\%$, p is calculated to be 0.009948187 and each scenario is assigned their corresponding probabilities $p\lambda^{j-1}$ with the most recent data having the highest probability.

Then, ΔV_j are sorted in ascending order, *i.e.*, from the largest loss to the largest gain. With the values sorted in ascending order, the cumulative probability for each scenario is calculated. The 1-cumulative probability for each scenario is also calculated. Then, the scenario where the 1-cumulative probability first decreases below 99% is determined.

Scenario	ΔMV	Probability	Cumulative	1 - Cumulative
141	(57,470.76)	0.002805793	0.0028058	0.997194207
146	(56,662.76)	0.002681784	0.0054876	0.994512423
139	(50,953.52)	0.002856987	0.0083446	0.991655436
142	(50,873.46)	0.002780541	0.0111251	0.988874895
154	(49,897.86)	0.00249467	0.0136198	0.986380225
143	(49,891.36)	0.002755516	0.0163753	0.98362471
145	(48,544.22)	0.00270614	0.0190814	0.98091857
144	(47,091.64)	0.002730716	0.0218121	0.978187854
5	(45,676.13)	0.009594858	0.031407	0.968592996
120	(43,720.05)	0.003392414	0.0347994	0.965200582

Figure 16: FX Forward: BRW Approach Calculations

Finally, the 1-day 99% VaR using the BRW approach is computed by interpolating between the ΔV_j that corresponds to the 99% 1-cumulative probability. The 1-day 99% TVaR using the BRW approach is obtained by taking the weighted average of the ΔV_j that have a 1-cumulative probability greater than 99%. Lastly, the VaR and TVaR are reported in PHP using the rate 47.06 USD/PHP.

	BRW	
λ	99.10%	
p	0.009948187	
1-day 99% VaR	50,905.85	USD
1-day 99% TVaR	53,935.61	USD
1-day 99% VaR	2,395,629.42	PHP
1-day 99% TVaR	2,538,210.01	PHP

Figure 17: FX Forward: VaR and TVaR (BRW Approach)

4.4 Results and Discussion

The 1-day 99% VaR and TVaR for the FX forward contract is obtained using the historical simulation approach and the BRW approach. On the one hand, using the historical simulation approach, the VaR is 2,505,343.32 PHP and the TVaR is 2,685,561.56 PHP. On the other hand, the BRW approach gives a VaR of 2,395,629.42 PHP and the TVaR is 2,538,210.01 PHP. It can be observed that the values for VaR and TVaR using the two approaches differ significantly. However, neither approach is better than the other.